Dowe, D.L. (1997). **The contribution of variance to utility functions**, pp 1-5, 1997 Australasian Meeting of the Econometric Society (ESAM'97), Volume 3 (Macroeconometrics and Finance), Melbourne, 2-4 July '97.

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The contribution of variance to utility functions

Abstract

In looking at the properties of the probability distribution of the return on an economic investment, economists often measure the dispersion of the distribution by its variance, σ^2. They then go on to value the investment as a mean-variance trade-off, $\mu - \beta\sigma^2$ for some (arbitrarily chosen) constant β, which is usually greater than 0. This mean-variance trade-off underpins the Capital Asset Pricing Model (CAPM), which we present briefly below, as well as many economic valuation models.

We show below that this mean-variance valuation can lead to situations where two distributions X and Y can satisfy for all c $Pr(X < c) \geq Pr(Y < c)$, i.e., that Y stochastically dominates X, but give $v(X) = \mu_X - \beta\sigma_X^2 > \mu_Y - \beta\sigma_Y^2 = v(Y)$.

Furthermore, Y can be taken to be such that $Pr(Y < 0) = 0$ but $v(Y) = -\infty$. Hence, we can conclude an inherent limitation in mean-variance trade-off models.

We then go on to show that if we are to value a distribution, X, by a mean-dispersion trade-off where our dispersion is given by $\int_{-\infty}^{\infty} f_X(x).|x - \mu_X|^r \, dx$, then the only sensible value for r is 1.

JEL Classification numbers: G12, G11

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Lemma 1: Let $\varepsilon < \frac{1}{4}$ be arbitrarily small.

Let X be defined so that $f_X(x) = \exp\{-\frac{1}{2}.((x-1)/\varepsilon)^2\}$, and

let Y be defined so that $Pr(Y \leq 2) = 0$ and $f_Y(y) = 8y^{-3}$ for $y > 2$.

Then $\mu_X = 1$, $\sigma_X = \varepsilon$, $\mu_Y = 4$, $\sigma_Y = \infty$.

$\forall c\; Pr(X < c) \geq Pr(Y < c)$, i.e., Y stochastically dominates X. but $v(X) = 1 - \beta\varepsilon^2 > -\infty = 4 - \beta\infty = v(Y)$.

Lemma 1 above accents the shortcomings of mean-variance modelling.
Lemma 2 below takes this result a bit further, telling us that if we measure dispersion using the expected value of any power $r > 1$ of the expected distance from the mean then the model will still be inappropriate.

Lemma 2: Let $r > 1$, and let Y be defined so that $Pr(Y < 2) = 0$ and $f_Y(y) = 2^r r y^{-1-r}$ for $y \geq 2$.

Then $\mu_Y = 2r/(r-1)$, $\sigma_Y = \infty$ and, although y is always positive, $v(Y) = -\infty$.

We have established the problems of using a power $r > 1$.
Lemma 3 predictably tells us that $r < 1$ also has problems.

Lemma 3: Let $r < 1$, $Pr(Y < 2) = 0$ and $f_Y(y) = 2y^{-2}$ for $y \geq 2$.

Then, for all constants c, $\int_{-\infty}^{\infty} f_Y(y).|y-c|^r \, dy$ is finite, but $\mu_Y = \int_{-\infty}^{\infty} f_Y(y).y \, dy = \infty$.

Lemma 4 serves the purpose of helping us derive Lemma 5.

Lemma 4: If c is a real number such that either $E(Y|Y \geq c)$ is finite or $E(Y|Y \leq c)$ is finite, then

$$\int_{-\infty}^{\infty} f_Y(y).|y-c| \, dy = E(Y|Y \geq c).Pr(Y \geq c) - E(Y|Y \leq c).Pr(Y \leq c)$$

Lemma 5: If $\forall c$ $\int_{-\infty}^{\infty} f_Y(y).|y-c| \, dy$ exists and is finite, then μ_Y exists and is finite.

Proof (of Lemma 5):
If the premise conditions hold, then it follows from Lemma 4 that $E(Y|Y \geq c).P(Y \geq c)$ and $E(Y|Y \leq c).P(Y \leq c)$ both exist and are finite. It then follows that
$E(Y|Y \geq c).Pr(Y \geq c) + E(Y|Y \leq c).Pr(Y \leq c)$ exists and is finite.
This equals μ_Y. Q.E.D.

Lemmas 4 and 5 tell us that, unless $E(Y|Y \geq c) = \infty$ and $E(Y|Y \leq c) = -\infty$ for every point c in the distribution, then using the absolute deviation to measure dispersion works perfectly in the sense that in this case, the mean is finite if and only if the dispersion is finite.

We thus conclude that if we value a distribution, X, by a mean-dispersion trade-off where our dispersion is given by

$\int_{-\infty}^{\infty} f_X(x).|x-\mu_X|^r \, dx$, then the only sensible value for r is 1.

This should say that the only sensible values are $r \leq 1$. (Unfortunately, the editors printed an earlier draft version of the paper.)

We also recall from Lemmas 1 and 2 that, with $r > 1$, it is possible to have a distribution with finite mean but infinite dispersion; and that it is thus possible to have two distributions X and Y such that $\forall c$ $Pr(X < c) \geq Pr(Y < c)$ but $V(X) = \mu_X - \beta\sigma_X^2 > -\infty = \mu_Y - \beta\infty = v(Y)$ for any $\beta > 0$.

With $r = 1$, this situation cannot occur, as we see from our main theorem, Theorem 6.

Theorem 6:

Suppose $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ and $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1$ and that $\forall c \int_{-\infty}^{c} f_X(x)\,dx \geq \int_{-\infty}^{c} f_Y(y)\,dy$.

Letting $d_Y = \int_{-\infty}^{\infty} f_Y(y).|y-\mu_Y|\,d_Y$ and $d_X = \int_{-\infty}^{\infty} f_X(x).|x-\mu_X|\,dx$,

then $\mu_Y \geq \mu_X$ and $d_Y \leq 2(\mu_Y - \mu_X) + d_X$.

Proof (of Theorem 6):

$$d_Y = 2\int_{-\infty}^{\mu_Y} f_Y(y)(\mu_Y - y)\,dy \qquad \text{since } \int_{-\infty}^{\mu_Y} f_Y(y).(\mu_Y - y)\,dy = \int_{\mu_Y}^{\infty} f_Y(y).(y-\mu_Y)\,dy$$

$$\leq 2\int_{-\infty}^{\mu_Y} f_X(x)(\mu_Y - x)\,dx \quad \text{by the inequality } \forall c \text{ applied for } c = \mu_Y$$

$$= 2(\mu_Y - \mu_X).\int_{-\infty}^{\mu_Y} f_X(x)\,dx + 2\int_{-\infty}^{\mu_X} f_X(x).(\mu_X - x)\,dx + 2\int_{\mu_X}^{\mu_Y} f_X(x).(\mu_X - x)\,dx$$

$$\leq 2(\mu_Y - \mu_X) + d_X \qquad \text{Q.E.D.}$$

Note: In case one tries to justify using the variance to measure the dispersion by the arguing that most distributions either are or approximate being normal, let us observe the following:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2.e^{-\frac{x^2}{2\sigma^2}}\,dx = \sigma^2 \quad \text{and} \quad \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} |x|.e^{-\frac{x^2}{2\sigma^2}}\,dx = \sqrt{\frac{2}{\pi}}\,\sigma .$$

Using the mean absolute difference to measure the dispersion of a normal distribution gives us a constant multiplied by the standard deviation.

Now, the Capital Asset Pricing Model CAPM (see, e.g. Oxelheim and Wihlborg, 1987, pp20-29), states that if R_F is the return on a zero-variance (risk-free) investment, $E[R_M]$ is the expected return on a balanced market portfolio and $E[R_j]$ is the expected return on commodity j, then

$E[R_j] - R_F = \beta(E[R_M] - R_F)$, where

$$\beta_j = cov(R_j, R_M)/\sigma_M^2 = \sigma_{jM}/\sigma_{MM}$$

$$= E((x_j - \mu_j)(x_M - \mu_M))/E((x_M - \mu_M)^2)$$

$$= (E(x_j x_M) - \mu_j \mu_M)/E\left((x_M - \mu_M)^2\right)$$

We have already shown that the assumption that distributions can be valued using a mean-variance trade-off, an assumption upon which CAPM depends, is flawed.

Let $f(R_M) = 3.03 \,.\, 2^{3.03}/R_{M^{3.03}}$ $R_M \geq 2$ $Pr(R_M < 2) = 0$

and let $f(R_j) = 2.02 \,.\, 2^{2.02}/R_j^{2.02}$ $R_j \geq 2$ $Pr(R_j < 2) = 0$.

Then $\mu_M < \mu_j < \infty$ and $\sigma_M^2 < \infty$, but $cov(R_j, R_M) = \infty$.

Acknowledgements:
This paper has gained from discussions with Keith McLaren and Chris Wallace.

References:

Brealey, R. and S. Myers, Principles of Corporate Finance, McGraw-Hill, Tokyo, 1984.

Lorie, J. and R. Brearley, Modern Developments in Investment Management, Dryden Press, Illinois, 1978.

Oxelheim, L. and C. Wihlborg, Macroeconomic Uncertainty - International Risks and Opportunities for the Corporation, Wiley, New York, 1987.